A Generalized Cross Ratio

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Abstract. In one complex variable, the cross ratio is a well-known quantity associated with four given points in the complex plane that remains invariant under linear fractional maps. In particular, if one knows where three points in the complex plane are mapped under a linear fractional map, one can use this invariance to explicitly determine the map and to show that linear fractional maps are 3-transitive. In this paper, we define a generalized cross ratio and determine some of its basic properties. In particular, we determine which hypotheses must be made to guarantee that our generalized cross ratio is well defined. We thus obtain a class of maps that obey similar transitivity properties as in one complex dimension, under some more restrictive conditions.

1. Background.

In 1872, Felix Klein famously introduced a point of view regarding what geometry should be about [**3**]. Known as the Erlangen program, Klein viewed geometry as a study of invariants under group transformations. An important and robust example of such an invariant quantity is given by the cross ratio.

Given four finite distinct points z_1, z_2, z_3, and z_4 in the complex plane, the cross ratio is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{1.1}$$

If $z_i = \infty$ for some $i = 1, 2, 3, 4$ then we cancel the terms with z_i. For example, if $z_1 = \infty$, then

$$(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_2 - z_3)}.$$

With this definition, we may view the cross ratio as a function of z given by (z, z_2, z_3, z_4). Recall that a linear fractional map, also known as a Mobius transformation, is defined as

$$f(z) = \frac{az + b}{cz + d}$$

where the coefficients a, b, c, and d are complex numbers such that $ad - bc \neq 0$ (otherwise $f(z)$ is a constant function).

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It is well known that the cross ratio is preserved under linear fractional maps. Thus if ϕ is a linear fractional map in one complex variable and $\phi(z_i) = w_i$ for $i = 1, 2, 3$ then

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

where $w = \phi(z)$. In such a case one may solve for w to determine the map explicitly.

2. Linear Fractional Maps in Several Complex Variables.

To define the cross ratio in several complex variables, we must first say what it means to be a linear fractional map in $\mathbb{C}^N$ for $N > 1$. Our approach to generalization will utilize homogeneous coordinates. We associate the point $z = (z_1', z_2)$ where $z_1' \in \mathbb{C}^N$ and $z_2 \in \mathbb{C}$, $z \neq 0$, with the point $\frac{z_1'}{z_2} \in \mathbb{C}^N$. This associated space is known as the complex projective space $\mathbf{CP}^N$. We will write $z \sim u$ if $z \in \mathbb{C}^N$ is the point associated with $u \in \mathbf{CP}^N$. In $\mathbb{C}$, one may show that linear fractional maps are precisely the linear transformations acting on homogeneous coordinates in $\mathbb{C}^2$. We take the perspective that linear fractional maps in $\mathbb{C}^N$ should be defined likewise.

For vectors $\begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ and $\begin{pmatrix} w_1' \\ w_2 \end{pmatrix}$ in $\mathbb{C}^{N+1}$ with $z_2, w_2 \in \mathbb{C}$, we consider a linear transformation in $\mathbb{C}^{N+1}$ represented by a complex $(N+1) \times (N+1)$ matrix as

$$\begin{pmatrix} w_1' \\ w_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \begin{pmatrix} z_1' \\ z_2 \end{pmatrix} = \begin{pmatrix} \langle a_1, \overline{z_1'} \rangle + b_1 z_2 \\ \vdots \\ \langle a_N, \overline{z_1'} \rangle + b_N z_2 \\ \langle z_1', C \rangle + D z_2 \end{pmatrix}$$

where A is an $N \times N$ matrix with rows denoted by a_i for $i = 1, ..., N$, $B = \begin{pmatrix} b_1 & \cdots & b_N \end{pmatrix}^T$ and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, C^* represents the conjugate transpose of C and $\langle \cdot, \cdot \rangle$ is the standard inner product.

If we now think of our vectors $\begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ and $\begin{pmatrix} w_1' \\ w_2 \end{pmatrix}$ as homogeneous coordinates of z and w so that $z \sim \begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1' \\ w_2 \end{pmatrix}$ then we can associate the above linear transformation in $\mathbb{C}^{N+1}$ with the not necessarily linear transformation in $\mathbb{C}^N$ given by

$$\begin{aligned} w = \frac{w_1'}{w_2} &= \left(\frac{\langle a_1, \overline{z_1'} \rangle + b_1 z_2}{\langle z_1', C \rangle + D z_2}, ..., \frac{\langle a_N, \overline{z_1'} \rangle + b_N z_2}{\langle z_1', C \rangle + D z_2} \right) \\ &= \left(\frac{\langle a_1, \overline{\frac{z_1'}{z_2}} \rangle + b_1}{\langle \frac{z_1'}{z_2}, C \rangle + D}, ..., \frac{\langle a_N, \overline{\frac{z_1'}{z_2}} \rangle + b_N}{\langle \frac{z_1'}{z_2}, C \rangle + D} \right) \\ &= \left(\frac{\langle a_1, \overline{z} \rangle + b_1}{\langle z, C \rangle + D}, ..., \frac{\langle a_N, \overline{z} \rangle + b_N}{\langle z, C \rangle + D} \right) \\ &= \frac{Az + B}{\langle z, C \rangle + D}. \end{aligned}$$

This is precisely the definition given by Cowen and MacCluer [**1**] and is the one we will adopt.

DEFINITION 2.1. We say ϕ is a linear fractional map in $\mathbb{C}^N$ if

$$\phi(z) = \frac{Az + B}{\langle z, C\rangle + D}. \tag{2.1}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

We define the associated matrix m_ϕ of the linear fractional map $\phi(z) = \frac{Az+B}{\langle z,C\rangle + D}$ to be given by

$$m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}.$$

If $\phi(z) = w$ and the point $u \in \mathbb{C}^{N+1}$ is associated with z, then $v = m_\phi u$ is associated with the point w and vice versa. A routine calculation also shows that $m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2}$ and $m_{\phi^{-1}} = (m_\phi)^{-1}$. That is, if $z \sim u$, then $\phi(z) \sim m_\phi u$. Thus, we can toggle back and forth between the complex space $\mathbb{C}^N$ and $\mathbf{CP}^N$ at our convenience.

3. The Cross Ratio in Several Complex Variables.

While generalized cross ratios have been defined previously in terms of the Schwarzian derivative [**2**] and from the perspective of symplectic geometry [**5**], this paper takes a different perspective that is more tractable and attempts to be in the elementary spirit of the traditional cross ratio. For one complex variable, one may show that the cross ratio is invariant under the maps ϕ and m_ϕ. We would like the same to be true for the cross ratio in $\mathbb{C}^N$. The cross ratio can also be utilized to show that any three pairwise distinct points in $\mathbb{C}$ that are sent to another set of three pairwise distinct points uniquely determine a linear fractional map in $\mathbb{C}$. The proof of this uses the fact that one can utilize cross ratios to define a unique map that sends a pairwise distinct triple to the standardized points $(1, 0, \infty)$. Generalizing to higher dimensions, however, does not come without cost. Even for $N = 2$, one may recall that projective transformations map lines to lines. Without further hypotheses to avoid problems of collinearity, the task of mapping a pairwise distinct quadruple to four standardized points is hopeless. Treading carefully, we use these facts to motivate our definition of a cross ratio in $\mathbb{C}^N$ for $N > 1$. In particular we seek a minimal definition that will preserve the properties of the cross ratio just described. Since we have reserved subscripts for the components of a point in several variables, for the remainder of our discussion we choose to designate *superscripts* for distinct points. As we will only consider linear fractional maps, there should be no confusion about notation. Just remember, superscripts will not designate exponents but rather distinct points.

Thus, as subscripts represent coordinates (components of a vector) and superscripts distinguish distinct points in higher dimensions, we define the following quantity. If we let $u^i = (u^i_1, u^i_2, ..., u^i_{N+1})$ for $i = 1, ..., N + 1$ be elements of $\mathbb{C}^{N+1}$, then we define

$$[u^1, u^2, ..., u^{N+1}] = \det \begin{pmatrix} u_1^1 & u_1^2 & u_1^3 & \dots & u_1^{N+1} \\ u_2^1 & u_2^2 & u_2^3 & \dots & u_2^{N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{N+1}^1 & u_{N+1}^2 & u_{N+1}^3 & \dots & u_{N+1}^{N+1} \end{pmatrix}. \tag{3.1}$$

In order to motivate our general definition, we begin by defining a cross ratio in $\mathbb{C}^2$ and proceed to generalize to $\mathbb{C}^N$. In particular,

DEFINITION 3.1. Given five distinct points u^i for $i = 1, ..., 5$ in $\mathbf{CP}^2$, we define the cross ratio as

$$(u^1, u^2, u^3, u^4, u^5) = \frac{[u^1, u^3, u^5][u^2, u^4, u^5]}{[u^1, u^4, u^5][u^2, u^3, u^5]}.$$

Note that although our notation for the determinant given by Equation 3.1 is only well-defined for vectors in $\mathbb{C}^{N+1}$, the definition for the cross ratio *is* well-defined for points in $\mathbf{CP}^N$.

One motivation for choosing this particular cross ratio is because it "reduces" to our usual cross ratio using the appropriate coordinates. We will also find it useful when addressing questions of transitivity. Let's look at an example of how, under the right conditions, this definition "reduces" to the standard definition of the cross ratio. For the vectors $u^i = (u_1^i, u_2^i, u_3^i)$ with $i = 1, .., 4$, if we standardize coordinates so that $u_3^i = 1$, $z_1^i = u_1^i/u_3^i$, and $z_2^i = u_2^i/u_3^i$, we get vectors in $\mathbb{C}^2/\{0\}$ which we can interpret as $\mathbf{CP}^1$. We still have one extra vector u^5 though. However, since we let $u_3^i = 1$ for $i = 1, ..., 4$, why not choose u^5 to be a "point at infinity". In our context, this means choosing the final coordinate to equal 0. If we choose $u^5 = (0, 1, 0)$, which is *a* point on the line at infinity, we have

$$\begin{aligned}
(u^1, u^2, u^3, u^4, u^5) &= \frac{[u^1, u^3, u^5][u^2, u^4, u^5]}{[u^1, u^4, u^5][u^2, u^3, u^5]} \\
&= \frac{\det \begin{pmatrix} u_1^1 & u_1^3 & 0 \\ u_2^1 & u_2^3 & 1 \\ u_3^1 & u_3^3 & 0 \end{pmatrix} \det \begin{pmatrix} u_1^2 & u_1^4 & 0 \\ u_2^2 & u_2^4 & 1 \\ u_3^2 & u_3^4 & 0 \end{pmatrix}}{\det \begin{pmatrix} u_1^1 & u_1^4 & 0 \\ u_2^1 & u_2^4 & 1 \\ u_3^1 & u_3^4 & 0 \end{pmatrix} \det \begin{pmatrix} u_1^2 & u_1^3 & 0 \\ u_2^2 & u_2^3 & 1 \\ u_3^2 & u_3^3 & 0 \end{pmatrix}} = \frac{\det \begin{pmatrix} z_1^1 & z_1^3 & 0 \\ z_2^1 & z_2^3 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det \begin{pmatrix} z_1^2 & z_1^4 & 0 \\ z_2^2 & z_2^4 & 1 \\ 1 & 1 & 0 \end{pmatrix}}{\det \begin{pmatrix} z_1^1 & z_1^4 & 0 \\ z_2^1 & z_2^4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det \begin{pmatrix} z_1^2 & z_1^3 & 0 \\ z_2^2 & z_2^3 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \\
&= \frac{(z^1 - z^3)(z^2 - z^4)}{(z^1 - z^4)(z^2 - z^3)} = (z^1, z^2, z^3, z^4).
\end{aligned}$$

In $\mathbb{C}$, there are $4! = 24$ permutations of (z^1, z^2, z^3, z^4) of which 6 are distinct. In $\mathbb{C}^2$, we recognize that

$$(z^i, z^j, z^k, z^l, z^m) = (z^k, z^l, z^i, z^j, z^m) = (z^j, z^i, z^l, z^k, z^m) = (z^l, z^k, z^j, z^i, z^m)$$

and that no other permutation is equal to $(z^i, z^j, z^k, z^l, z^m)$. Thus, we have five ways to choose m and once m is chosen, we start with i (by the above equation we may, equivalently, start with any of i, j, k, l and have $3! = 6$ ways to choose the remaining three values which gives us $5 \times 6 = 30$ distinct permutations.

Let's see, explicitly, what this definition looks in like our usual complex variables when the z^i's for $i = 1, ..., 5$ are finite complex numbers where we are once again putting the notation for the distinct points in the superscript, leaving the subscripts for vector components. This means we set the third coordinate to 1 in each case. Thus we have

$$
\begin{aligned}
(u^1, u^2, u^3, u^4, u^5) &= \frac{[u^1, u^3, u^5][u^2, u^4, u^5]}{[u^1, u^4, u^5][u^2, u^3, u^5]} \\
&= \frac{\det\begin{pmatrix} u_1^1 & u_1^3 & u_1^5 \\ u_2^1 & u_2^3 & u_2^5 \\ u_3^1 & u_3^3 & u_3^5 \end{pmatrix} \det\begin{pmatrix} u_1^2 & u_1^4 & u_1^5 \\ u_2^2 & u_2^4 & u_2^5 \\ u_3^2 & u_3^4 & u_3^5 \end{pmatrix}}{\det\begin{pmatrix} u_1^1 & u_1^4 & u_1^5 \\ u_2^1 & u_2^4 & u_2^5 \\ u_3^1 & u_3^4 & u_3^5 \end{pmatrix} \det\begin{pmatrix} u_1^2 & u_1^3 & u_1^5 \\ u_2^2 & u_2^3 & u_2^5 \\ u_3^2 & u_3^3 & u_3^5 \end{pmatrix}} = \frac{\det\begin{pmatrix} z_1^1 & z_1^3 & z_1^5 \\ z_2^1 & z_2^3 & z_2^5 \\ 1 & 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^4 & z_1^5 \\ z_2^2 & z_2^4 & z_2^5 \\ 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} z_1^1 & z_1^4 & z_1^5 \\ z_2^1 & z_2^4 & z_2^5 \\ 1 & 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^3 & z_1^5 \\ z_2^2 & z_2^3 & z_2^5 \\ 1 & 1 & 1 \end{pmatrix}} \\
&= \frac{\left(z_1^1(z_2^3 - z_2^5) - z_1^3(z_2^1 - z_2^5) + z_1^5(z_2^1 - z_2^3)\right)\left(z_1^2(z_2^4 - z_2^5) - z_1^4(z_2^2 - z_2^5) + z_1^5(z_2^2 - z_2^4)\right)}{\left(z_1^1(z_2^4 - z_2^5) - z_1^4(z_2^1 - z_2^5) + z_1^5(z_2^1 - z_2^4)\right)\left(z_1^2(z_2^3 - z_2^5) - z_1^3(z_2^2 - z_2^5) + z_1^5(z_2^2 - z_2^3)\right)} \\
&= (z^1, z^2, z^3, z^4, z^5).
\end{aligned}
$$

This expression is definitely cleaner expressed in terms of determinants! As an exercsie for the reader, try writing down the expression when one of the z^i's is a point on the line at infinity.

Next we recall that by letting $z = z^1$ be a complex variable, the cross ratio in one complex variable becomes the linear fractional map given by

$$(z, z^2, z^3, z^4) = \frac{(z - z^3)(z^2 - z^4)}{(z - z^4)(z^2 - z^3)}.$$ [1]

Notice this cross ratio sends the points z^2, z^3, and z^4 to 1, 0, and ∞, respectively. We would like our generalized cross ratio to be able to recover this highlight in some way. In order to achieve this, we first have to extend our definition. After all, if we let $z = z^1 = (z_1, z_2)$, then the above expression for $(z^1, z^2, z^3, z^4, z^5)$ does not give us a linear fractional map as given in Definition 2.1. Such a map should map $\mathbb{C}^2$ to $\mathbb{C}^2$, not $\mathbb{C}^2$ to $\mathbb{C}$! Thus we define the cross ratio pair.

Definition 3.2. We define the cross ratio pair in $\mathbf{CP}^2$ as

$$(u^1, u^2, u^3, u^4, u^5)_2 = \left(\frac{[u^1, u^3, u^5][u^2, u^4, u^5]}{[u^1, u^4, u^5][u^2, u^3, u^5]}, \frac{[u^1, u^3, u^4][u^2, u^4, u^5]}{[u^1, u^4, u^5][u^2, u^3, u^4]} \right)$$

where we see that this defines a linear fractional map when the point associated with u^1 is a variable in $\mathbb{C}^2$.

Motivated by the above results, we define a generalized cross ratio. Although the notation is a bit more intimidating, this definition is nothing more than the natural extension of the two variable case.

[1] This is just Equation 1.1 with superscript notation instead.

Definition 3.3. Given $N+3$ distinct points u^i for $i = 1, ..., N+3$ in $\mathbf{CP}^N$, we define the cross ratio as

$$(u^1, u^2, u^3, u^4, u^5, ..., u^{N+3}) = \frac{[u^1, u^3, u^5, ..., u^{N+3}][u^2, u^4, u^5, ..., u^{N+3}]}{[u^1, u^4, u^5, ..., u^{N+3}][u^2, u^3, u^5, ..., u^{N+3}]}$$

where each ellipsis represents the sequence $\{u_k\}_{k=6}^{N+2}$.

To simplify the definition of the cross ratio N-tuple, we introduce new notation. We define $[u^i, u^j]_N^c$ by

$$[u^i, u^j]_N^c = \left[\{u^k\}_{k=1}^{N+3}\right] \setminus \{u_i, u_j\} = [u^1, ..., u^{i-1}, u^{i+1}, ..., u^{j-1}, u^{j+1}, ..., u^{N+3}].$$

This sets the stage for our general definition. If the notation appears intimidating, just remind yourself that it is the natural generalization of the definition of the cross ratio pair in $\mathbf{CP}^2$.

Definition 3.4. We define the cross ratio N-tuple to be given by

$$(u^1, u^2, ..., u^{N+2}, u^{N+3})_N = \left(\left\{ \frac{[u^2, u^i]_N^c [u^1, u^3]_N^c}{[u^2, u^3]_N^c [u^1, u^i]_N^c} \right\}_{i=4,...,N+3} \right)$$

where the curly brackets denote a sequence over $i = 4, ..., N+3$.

As in the two variable case, this defines a linear fractional map when the point associated with u^1 is a variable in $\mathbb{C}^N$.

We saw that we could "reduce" our cross ratio in two variables to the one variable definition. The next theorem tells us that this works in any dimension. The reasoning, again, is a natural extension of what was demonstrated in two variables.

Theorem 3.5. *Identifying coordinates* $z_j^i = \frac{u_j^i}{u_j^{N+1}}$ *for* $j = 1, ..., N$ *and* $u^{N+3} = (0, 0, ..., 0, 1, 0)^T$*, the cross ratio in* $\mathbb{C}^N$ *reduces to the cross ratio in* $\mathbb{C}^{N-1}$*.*

Proof. We use proof by induction. We have seen that our result is true for $N = 2$. Now suppose $N = k+1$ where $k > 1$, making the appropriate identifications, we have

$$(u^1, u^2, ..., u^{(k+1)+3}) = \frac{[u^1, u^3, u^5, ..., u^{k+4}][u^2, u^4, u^5, ..., u^{k+4}]}{[u^1, u^4, u^5, ..., u^{k+4}][u^2, u^3, u^5, ..., u^{k+4}]}$$

$$= \frac{\det\begin{pmatrix} u_1^1 & u_1^3 & u_1^5 & \dots & u_1^{k+4} \\ u_2^1 & u_2^3 & u_2^5 & \dots & u_2^{k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{k+2}^1 & u_{k+2}^3 & u_{k+2}^5 & \dots & u_{k+2}^{k+4} \end{pmatrix} \det\begin{pmatrix} u_1^2 & u_1^4 & u_1^5 & \dots & u_1^{k+4} \\ u_2^2 & u_2^4 & u_2^5 & \dots & u_2^{k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{k+2}^2 & u_{k+2}^4 & u_{k+2}^5 & \dots & u_{k+2}^{k+4} \end{pmatrix}}{\det\begin{pmatrix} u_1^1 & u_1^4 & u_1^5 & \dots & u_1^{k+4} \\ u_2^1 & u_2^4 & u_2^5 & \dots & u_2^{k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{k+2}^1 & u_{k+2}^4 & u_{k+2}^5 & \dots & u_{k+2}^{k+4} \end{pmatrix} \det\begin{pmatrix} u_1^2 & u_1^3 & u_1^5 & \dots & u_1^{k+4} \\ u_2^2 & u_2^3 & u_2^5 & \dots & u_2^{k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{k+2}^2 & u_{k+2}^3 & u_{k+2}^5 & \dots & u_{k+2}^{k+4} \end{pmatrix}}$$

$$= \frac{\det\begin{pmatrix} z_1^1 & z_1^3 & z_1^5 & \dots & 0 \\ z_2^1 & z_2^3 & z_2^5 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+1}^1 & z_{k+1}^3 & z_{k+1}^5 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^4 & z_1^5 & \dots & 0 \\ z_2^2 & z_2^4 & z_2^5 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+1}^2 & z_{k+1}^4 & z_{k+1}^5 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}}{\det\begin{pmatrix} z_1^1 & z_1^4 & z_1^5 & \dots & 0 \\ z_2^1 & z_2^4 & z_2^5 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+1}^1 & z_{k+1}^4 & z_{k+1}^5 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^3 & z_1^5 & \dots & 0 \\ z_2^2 & z_2^3 & z_2^5 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+1}^2 & z_{k+1}^3 & z_{k+1}^5 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}}$$

$$= \frac{\det\begin{pmatrix} z_1^1 & z_1^3 & z_1^5 & \dots & z_1^{k+3} \\ z_2^1 & z_2^3 & z_2^5 & \dots & z_2^{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^1 & z_k^3 & z_k^5 & \dots & z_k^{k+3} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^4 & z_1^5 & \dots & z_1^{k+3} \\ z_2^2 & z_2^4 & z_2^5 & \dots & z_2^{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^2 & z_k^4 & z_k^5 & \dots & z_k^{k+3} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}}{\det\begin{pmatrix} z_1^1 & z_1^4 & z_1^5 & \dots & z_1^{k+3} \\ z_2^1 & z_2^4 & z_2^5 & \dots & z_2^{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^1 & z_k^4 & z_k^5 & \dots & z_k^{k+3} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \det\begin{pmatrix} z_1^2 & z_1^3 & z_1^5 & \dots & z_1^{k+3} \\ z_2^2 & z_2^3 & z_2^5 & \dots & z_2^{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k^2 & z_k^3 & z_k^5 & \dots & z_k^{k+3} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}}$$

$$= (z^1, z^2, ..., z^{k+3}).$$

□

We next show that this definition satisfies the invariant properties we would expect a cross ratio to have. Recall that for a nonzero parameter λ, the points $(\lambda u, \lambda v)$ and (u, v) represent the same point in $\mathbf{CP}^1$ and are associated with the same point in the complex plane. Which representative we choose, however, doesn't matter.

THEOREM 3.6. *The cross ratio is independent of the representative chosen.*

PROOF. Given $N+3$ nonzero parameters $\{\lambda_i\}_{i=1,...,N+3}$, we have

$$\begin{aligned}
&(\lambda_1 u^1, \lambda_2 u^2, ..., \lambda_{N+2} u^{N+2}, \lambda_{N+3} u^{N+3}) \\
&= \frac{[\lambda_1 u^1, \lambda_3 u^3, \lambda_5 u^5, ..., \lambda_{N+3} u^{N+3}][\lambda_2 u^2, \lambda_4 u^4, \lambda_5 u^5, ..., \lambda_{N+3} u^{N+3}]}{[\lambda_1 u^1, \lambda_4 u^4, \lambda_5 u^5, ..., \lambda_{N+3} u^{N+3}][\lambda_2 u^2, \lambda_3 u^3, \lambda_5 u^5, ..., \lambda_{N+3} u^{N+3}]} \\
&= \frac{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5^2 ... \lambda_{N+3}^2 [u^1, u^3, u^5, ..., u^{N+3}][u^2, u^4, u^5, ..., u^{N+3}]}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5^2 ... \lambda_{N+3}^2 [u^1, u^4, u^5, ..., u^{N+3}][u^2, u^3, u^5, ..., u^{N+3}]} \\
&= \frac{[u^1, u^3, u^5, ..., u^{N+3}][u^2, u^4, u^5, ..., u^{N+3}]}{[u^1, u^4, u^5, ..., u^{N+3}][u^2, u^3, u^5, ..., u^{N+3}]} = (u^1, u^2, ..., u^{N+2}, u^{N+3})
\end{aligned}$$

□

By the same reasoning it is easy to see that the cross ratio N-tuple is also independent of the representative chosen.

We next show that this cross ratio is invariant under associated matrices of linear fractional maps.

THEOREM 3.7. *Let m_ϕ be the associated matrix to an invertible linear fractional map ϕ; that is,* $\det m_\phi \neq 0$. *Then the cross ratio is invariant under m_ϕ.*

PROOF. Note since $[Av_1, Av_2, ..., Av_N] = \det(A)[v_1, v_2, ..., v_N]$, we have

$$[m_\phi u^1, m_\phi u^2, ..., m_\phi u^k] = \det(m_\phi)[u^1, u^2, ..., u^k].$$

Thus

$$\begin{aligned}
&(m_\phi u^1, m_\phi u^2, ..., m_\phi u^{N+3}) \\
&= \frac{[m_\phi u^1, m_\phi u^3, m_\phi u^5, ..., m_\phi u^{N+3}][m_\phi u^2, m_\phi u^4, m_\phi u^5, ..., m_\phi u^{N+3}]}{[m_\phi u^1, m_\phi u^4, m_\phi u^5, ..., m_\phi u^{N+3}][m_\phi u^2, m_\phi u^3, m_\phi u^5, ..., m_\phi u^{N+3}]} \\
&= \frac{\det(m_\phi)^2 [u^1, u^3, u^5, ..., u^{N+3}][u^2, u^4, u^5, ..., u^{N+3}]}{\det(m_\phi)^2 [u^1, u^4, u^5, ..., u^{N+3}][u^2, u^3, u^5, ..., u^{N+3}]} \\
&= \frac{[u^1, u^3, u^5, ..., u^{N+3}][u^2, u^4, u^5, ..., u^{N+3}]}{[u^1, u^4, u^5, ..., u^{N+3}][u^2, u^3, u^5, ..., u^{N+3}]} = (u^1, u^2, ..., u^{N+2}, u^{N+3})
\end{aligned}$$

where we can divide by $\det(m_\phi)$ since we presume m_ϕ is invertible. □

4. Transitivity of linear fractional maps in several complex variables.

It is well known that linear fractional maps acting on $\mathbb{C}$ are 3-transitive but not 4-transitive. That is, for six points z^1, z^2, z^3, w^1, w^2, and w^3 in $\mathbb{C}$ where the z^i's are pairwise distinct as are the w^i's, there is a unique linear fractional map χ such that $\chi(z^i) = w^i$ for $i = 1, 2, 3$. The usual way to achieve this result is to send the triple (z^1, z^2, z^3) to the standardized points $(1, 0, \infty)$ where ∞ represents the "north pole" on the Riemann sphere. In homogeneous coordinates, these standardized points are associated with $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\0\end{pmatrix}$, respectively. Thus, for the cross ratio as a function of z with associated point u given by $\phi(z) = \frac{(z-z^2)(z^1-z^3)}{(z-z^3)(z^1-z^2)}$, we may write the point associated with $\phi(z)$ as

$$m_\phi u = \begin{pmatrix} (z-z^2)(z^1-z^3) \\ (z-z^3)(z^1-z^2) \end{pmatrix}.$$

It is clear that this sends our triple (z^1, z^2, z^3) to the points $(1, 0, \infty)$, respectively. Note that taking two distinct points z^1 and z^2 in $\mathbb{C}$ implies that the associated points in $\mathbf{CP}^1$, when seen as vectors in $\mathbb{C}^2$, are linearly independent. However, it is not true that taking three distinct points in $\mathbb{C}^N$ implies the associated points, when seen as vectors in $\mathbb{C}^{N+1}$, will be linearly independent. They could be collinear, for example. This potential obstruction will necessitate an additional assumption of linear independence that was not needed in $\mathbb{C}$.

We would like our cross ratio N-tuple to be a generalized formula for a multi-dimensional linear fractional map that maps $N+2$ points to certain pre-specified points analogous to the triple $(1, 0, \infty)$. To achieve this, we consider an $N+2$-tuple in which $N+1$ of the points have associated points in $\mathbf{CP}^N$ which, when seen as vectors in $\mathbb{C}^{N+1}$, are linearly independent. We then send these points to the standard basis vectors $\{e_i\}_{i=1}^{N+1}$ and the remaining point is sent to $\sum_{i=1}^{N+1} e_i$. To begin, we see how to do this in $\mathbb{C}^2$ and generalize.

Let $u^i \in \mathbf{CP}^2$ be associated with $z^i \in \mathbb{C}^2$ for $i = 2, ..., 5$ and u associated with z. Define the linear fractional map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\begin{aligned}\phi(z) &= (z, z^2, z^3, z^4, z^5)_2 = (u, u^2, u^3, u^4, u^5)_2 \\ &= \left(\frac{[u, u^3, u^5][u^2, u^4, u^5][u^2, u^3, u^4]}{[u, u^4, u^5][u^2, u^3, u^4][u^2, u^3, u^5]}, \frac{[u, u^3, u^4][u^2, u^3, u^5][u^2, u^4, u^5]}{[u, u^4, u^5][u^2, u^3, u^4][u^2, u^3, u^5]} \right).\end{aligned}$$

If the point $u \in \mathbf{CP}^2$ is associated with $z \in \mathbb{C}^2$, then the point $v \in \mathbf{CP}^2$ associated with $w = \phi(z) \in \mathbb{C}^2$ is given by

$$v = m_\phi u = \begin{pmatrix} [u, u^3, u^5][u^2, u^4, u^5][u^2, u^3, u^4] \\ [u, u^3, u^4][u^2, u^3, u^5][u^2, u^4, u^5] \\ [u, u^4, u^5][u^2, u^3, u^4][u^2, u^3, u^5] \end{pmatrix}. \tag{4.1}$$

We recall that $[u^i, u^j, u^k] = 0$ when any of i, j, k are equal or, more generally, when there is linear dependence. To keep the cross ratio map given above well-defined, we introduce the following hypothesis.

Definition 4.1. Let the set $\{u^i\}_{i=3}^{N+3}$ in $\mathbb{C}^{N+1}$ form a linearly independent set. We make the assumption that $u^2 = \sum_{i=0}^{N} \alpha_i u^{i+3}$ where $\{\alpha_i\}_{i=0}^{N}$ are all non-zero complex numbers. We will call this *the independence hypothesis*.

Considering the N-dimensional space $\left(\{\alpha_i\}_{i=0}^{N}\right)$, the independence hypothesis asks only that we omit a set of measure zero. Additionally, If a set of points in $\mathbf{CP}^N$, when viewed as vectors in $\mathbb{C}^{N+1}$ satisfies the independence hypothesis, we say the set of points in $\mathbf{CP}^N$ satisfies the independence hypothesis.

With this in mind, it is clear to see that we have $m_\phi u^2 = (1, 1, 1)$, $m_\phi u^3 = (0, 0, 1)$, $m_\phi u^4 = (1, 0, 0)$, and $m_\phi u^5 = (0, 1, 0)$ as desired where $(1, 1, 1)$ is associated with $(1, 1)$, $(0, 0, 1)$ with $(0, 0)$, and vectors such as $(1, 0, 0)$, which we will denote by $e_{1,\infty}$, and $(0, 1, 0)$, which we will denote by $e_{2,\infty}$, correspond to the points in the hyperplane at infinity that are tangent to $(0, 1)$ and $(1, 0)$, respectively. In general, our goal will then be to map $N+2$ points associated with $N+2$ points in $\mathbf{CP}^N$ which satisfy the independence hypothesis to a set of $N+2$ standardized points in $\mathbb{C}^N$ with associated points $\sum_{k=1}^{N+1} e_k$ and $\{e_k\}_{k=1}^{N+1}$ where e_i is the vector that is zero on every coordinate except the ith coordinate where it takes the value of 1.

THEOREM 4.2. *Given $N+2$ distinct elements z^2, z^3,...,z^{N+3} in $\mathbb{C}^N$ with respective associated points u^2, u^3,...,u^{N+3} in $\mathbf{CP}^N$ that satisfy the independence hypothesis, there exists a unique linear fractional map ϕ such that $m_\phi u^2 = (1,1,...,1) = \sum_{k=1}^{N+1} e_k$, $m_\phi u^3 = e_{N+1}$, $m_\phi u^4 = e_1$, $m_\phi u^5 = e_2$, and $m_\phi u^i = e_{i-3}$ for $i > 5$.*

PROOF. Define the linear fractional map ϕ by

$$\begin{aligned}\phi(z) &= (z^1, z^2, ..., z^{N+2}, z^{N+3})_N \\ &= (u^1, u^2, ..., u^{N+2}, u^{N+3})_N = \left(\left\{ \frac{[u^2, u^i]_N^c [u, u^3]_N^c}{[u^2, u^3]_N^c [u, u^i]_N^c} \right\}_{i=4,...,N+3} \right).\end{aligned}$$

We may then write the point associated with $\phi(z)$ as

$$v = m_\phi u = \begin{pmatrix} \frac{[u^2,u^4]_N^c}{[u,u^4]_N^c} & \frac{[u^2,u^5]_N^c}{[u,u^5]_N^c} & \cdots & \frac{[u^2,u^3]_N^c}{[u,u^3]_N^c} \end{pmatrix}^T.$$

We recall that $[u^i, u^j]_N^c = [u, ..., u^{i-1}, u^{i+1}, ..., u^{j-1}, u^{j+1}, ...u^{N+3}] = 0$ when $u = u^k$ for some integer $1 < k < N+3$, $k \neq i, j$. In particular, note that none of the denominators in the entries of v are zero. Thus we have $m_\phi u^2 = \sum_{k=1}^{N+1} e_k$, $m_\phi u^3 = e_{N+1}$, $m_\phi u^4 = e_1$, $m_\phi u^5 = e_2$, and $m_\phi u^k = e_{k-3}$.

We next demonstrate uniqueness. Suppose there are two linear fractional maps ϕ and ψ such that the associated matrices m_ϕ and m_ψ send each u^i as above. Then the associated matrix $M = m_\phi m_\psi^{-1}$ fixes e_i and $p = \sum_{i=1}^N e_i$. Denote M by

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \dots & a_{NN} \end{pmatrix}.$$

Then $Me_i = e_i$ implies $a_{ij} = 0$ for $i \neq j$ and $Mp = p$ implies $a_{ii} = a_{jj}$ for all $1 \leq i, j \leq N$. We conclude that $M = a_{11} I$ where I is the identity which implies $m_\phi = a_{11} m_\psi$ and thus $\phi = \psi$. □

To generalize in several variables, we use transitivity in a more restricted sense. In particular, we make the additional assumption that the initial data points in $\mathbb{C}^N$ have associated points in $\mathbf{CP}^N$ that satisfy the independence hypothesis. We next show that, in this sense of transitivity, linear fractional maps in $\mathbb{C}^N$ are $(N+2)$-transitive but not $(N+3)$-transitive.

THEOREM 4.3. *Given $N+2$ pairwise distinct points $\{z^i\}_{i=2}^{N+3}$ in $\mathbb{C}^N$ with associated points $\{u^i\}_{i=2}^{N+3}$ and $N+2$ pairwise distinct points $\{w^i\}_{i=2}^{N+3}$ in $\mathbb{C}^N$ with associated points $\{v^i\}_{i=2}^{N+3}$ such that the sets $\{u^i\}_{i=2}^{N+3}$ and $\{v^i\}_{i=2}^{N+3}$ satisfy the independence hypothesis, there exists a unique linear fractional map $\chi(z)$ such that $\chi(z^i) = w^i$ for $i = 2, ..., N+3$.*

PROOF. By Theorem 4.2, there exist maps ϕ and ψ such that $m_\phi u^2 = m_\psi v^2 = \sum_{k=1}^{N+1} e_k$, $m_\phi u^3 = m_\psi v^3 = e_{N+1}$, $m_\phi u^4 = m_\psi v^4 = e_1$, $m_\phi u^5 = m_\psi v^5 = e_2$, and $m_\phi u^k = m_\psi v^k = e_{k-3}$ otherwise. Define $m_\gamma = m_\psi^{-1} m_\phi$. This sends u^i to v^i for $i = 2, ..., N+3$. To show uniqueness, suppose m_σ maps u^i to v^i for $i = 2, ..., N+3$. Then m_ϕ and $m_\psi m_\sigma$ each send u^i to the same values for $i = 2, ..., N+3$. Thus by Theorem 4, we have $m_\phi = m_\psi m_\sigma$ which implies $m_\gamma = m_\psi^{-1} m_\phi = m_\sigma$ from which we conclude that $\gamma = \sigma$ as desired. □

THEOREM 4.4. *If m_ϕ fixes $N+2$ distinct points $\{u^i\}_1^{N+2}$ satisfying the independence hypothesis, then ϕ is the identity.*

PROOF. Suppose that m_ϕ fixes $N+2$ distinct points $\{u^i\}_1^{N+2}$ satisfying the independence hypothesis, then, since the identity also fixes these points, by Theorem 5 we have that m_ϕ is equal to the identity and thus ϕ is the identity. □

We can thus conclude that linear fractional maps in $\mathbb{C}^N$ are $(N+2)$-transitive but they are not $(N+3)$-transitive.

5. More to explore.

The author invites the reader to investigate this generalized definition further. For example, in one variable, we have a well-known criterion to determine when the cross ratio is real.

THEOREM 5.1. *The cross ratio (z^1, z^2, z^3, z^4) is real if and only if the four points lie on a circle (where we let a line be a circle of infinite radius).*

PROOF. Consider the cross ratio as a function of the first argument, $\phi(z) = (z, z^1, z^2, z^3)$. We saw that ϕ maps the triple (z^1, z^2, z^3) to $(1, 0, \infty)$, respectively. Since linear fractional maps send circles to circles, we conclude that ϕ sends our circle containing the triple (z^1, z^2, z^3) to the extended real line. Now consider the map $\phi^{-1}(\phi(z)) = z$. It is known that linear fractional maps are bijective and thus $\phi(z)$ is real if and only if $z = \phi^{-1}(\phi(z))$ lies on the circle containing our triple (z^1, z^2, z^3) which is the image of the extended real line under our map ϕ^{-1}. □

In order for this theorem to extend to our generalized cross ratio N-tuple, it will take an upgrade of some geometric facts about these generalized maps. We appeal to results by Cowen and MacCluer (see [**1**], Section 3 for more details.). In particular, one finds more flexibility when working in higher dimensions. The appropriate generalization of circles in this situation is to consider *ellipsoids*, where an ellipsoid is a translate of the image of the unit ball under an invertible complex linear transformation. We recall Theorem 6 from [1].

THEOREM 5.2. *If ϕ is a one-to-one linear fractional map defined on a ball $\overline{\mathbb{B}}$ in $\mathbb{C}^N$, then $\phi(\mathbb{B})$ is an ellipsoid.*

PROOF. See [1], section 3. □

This brings us to the following open question, left for the reader to demonstrate.

QUESTION 5.3. Is there an analogue to Theorem 5.1 for the generalized cross ratio if we relax the presumption of points lying on the boundary of a sphere and instead use the boundary of an ellipsoid?

In some form or another, the cross ratio and its special invariant properties have been examined and celebrated since antiquity. Now that we have a new perspective to talk about this quantity in higher dimensions, the author challenges the reader to explore what other similarities or differences the generalized version of this classical quantity has.

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